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# On Bivariate Cardinal Interpolation by Shifted Splines on a Three-Direction Mesh

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## 1. INTRODUCTION

In this paper, we study cardinal interpolation of bounded data by integer translates of shifted symmetric box splines based on a three-direction mesh. To set notation, let $M_n$ denote the centred bivariate box spline corresponding to the directions $(1, 0)$, $(0, 1)$, $(1, 1)$, each occurring with equal multiplicity $n$ (see [1] for the relevant definition). For any vector $\alpha := (\alpha_1, \alpha_2)$, we denote by $M_{n,\alpha}$ the function $M_n(\cdot + \alpha)$. Let $P_{n,\alpha}$ stand for the *characteristic polynomial* given by

$$P_{n,\alpha}(x) := \sum_{j \in \mathbf{Z}^2} M_{n,\alpha}(j) e^{-ijx} \qquad x \in \mathbf{R}^2. \tag{1.1}$$

By virtue of the Poisson summation formula [4, p. 129], we have

$$\begin{aligned} P_{n,\alpha}(x) &= \sum_{j \in \mathbf{Z}^2} \hat{M}_{n,\alpha}(x + 2\pi j) \\ &= \sum_{j \in \mathbf{Z}^2} \hat{M}_n(x + 2\pi j) e^{i\alpha(x + 2\pi j)}, \end{aligned} \tag{1.2}$$

where $\hat{M}_n$ denotes the Fourier transform of $M_n$. Let us recall here that $\hat{M}_n$ is given by (cf. [1])

$$\hat{M}_n(t_1, t_2) = [S(t_1) S(t_2) S(t_1 + t_2)]^n \tag{1.3}$$

with

$$S(t) := \left[\frac{\sin(t/2)}{(t/2)}\right]. \tag{1.4}$$



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## 2. Cardinal Interpolation with $M_{n,\alpha}$

Let $\psi: \mathbf{R}^2 \to \mathbf{R}$ be a compactly supported continuous function. The cardinal interpolation problem (CIP) with $\psi$ is said to be *correct* if to each bounded real-valued function $f$ defined on $\mathbf{R}^2$, there corresponds a unique bounded sequence $\{a_j : j \in \mathbf{Z}^2\}$ such that $\sum_{j \in \mathbf{Z}^2} a_j \psi(\cdot - j)$ agrees with $f$ on $\mathbf{Z}^2$. In the case when $\psi$ is a box-spline based on a three-direction $[(1, 0), (0, 1), (1, 1)]$ mesh, this problem was first studied by C. de Boor, K. Höllig, and S. Riemenschneider in [1]. Therein, it was shown that cardinal interpolation with such a box-spline $M$ is always correct for any arbitrary selection of positive mutiplicities in each of its mesh directions. In particular, this is certainly true for the choice $M = M_n$. The focus of interest in the present paper is a "shifted" version of this result, namely, the correctness of the CIP with $M_{n,\alpha}$. Since $M_{n,\alpha}$ is also a continuous function with compact support, we have the following useful necessary and sufficient condition.

THEOREM 2.1. *Cardinal interpolation with $M_{n,\alpha}$ is correct if and only if $P_{n,\alpha}(x)$ does not vanish.*

*Proof.* See [1, p. 536, Theorem 2].

## 3. Symmetries of $M_{n,\alpha}$ and $P_{n,\alpha}$

We now deduce some symmetries of $M_{n,\alpha}$ and $P_{n,\alpha}$ which serve to make any further analysis less onerous. This study follows the tone set by de Boor *et al.* [1]. As the details of the proofs of our assertions are similar to those in [1], they will be omitted here.

Following [1], let $\mathbf{A}$ denote the group of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

that is isomorphic to $\mathbf{S}_3$ (the group of permutations on 3 indices). Then, for any $A \in \mathbf{A}$ and $x \in \mathbf{R}^2$, there hold

$$M_{n, \pm A\alpha}(\pm Ax) = M_{n,\alpha}(x), \tag{3.1}$$

$$\hat{M}_{n, \pm A\alpha}(x) = \hat{M}_{n,\alpha}(\pm A^* x), \tag{3.2}$$

$$P_{n, \pm A\alpha}(x) = P_{n,\alpha}(\pm A^* x), \tag{3.3}$$

where $A^*$ denotes the transpose of $A$. As for the relevance and importance of studying the role of $\mathbf{A}$ and $\mathbf{A}^* := \{A^* : A \in \mathbf{A}\}$, the reader may wish to consult [1] and [2].

## 4. Correctness of the CIP

We begin this central section by introducing some more notation. For convenience, we set $x := (2\pi u, 2\pi v)$ and $\theta := (\theta_1, \theta_2) := (2\pi\alpha_1, 2\pi\alpha_2)$. Let $\Omega$ and $\Lambda$ denote the interiors of the regions in Figs. 4.1 and 4.2, respectively. Let $\tilde{\Omega}$ represent the interior of the quadrilateral whose vertices are

$$(0, 0), \qquad (\tfrac{1}{2}, 0), \qquad (\tfrac{1}{3}, \tfrac{1}{3}), \qquad \text{and} \qquad (0, \tfrac{1}{2})$$

(see Fig. 4.1) and let the (closed) triangle (see Fig. 4.3) with vertices at

$$(0, 0), \qquad (\tfrac{1}{2}, 0), \qquad \text{and} \qquad (\tfrac{1}{3}, \tfrac{1}{3})$$

be denoted by $\Delta$. We further subdivide $\Delta$ into two parts, $\Delta_1$ and $\Delta_2$ (Fig. 4.3), where $\Delta_1$ is the quadrilateral formed by the vertices

$$(0, 0), \qquad (\tfrac{19}{48}, 0), \qquad (\tfrac{1}{3}, \tfrac{1}{6}), \qquad \text{and} \qquad (\tfrac{5}{18}, \tfrac{5}{18})$$

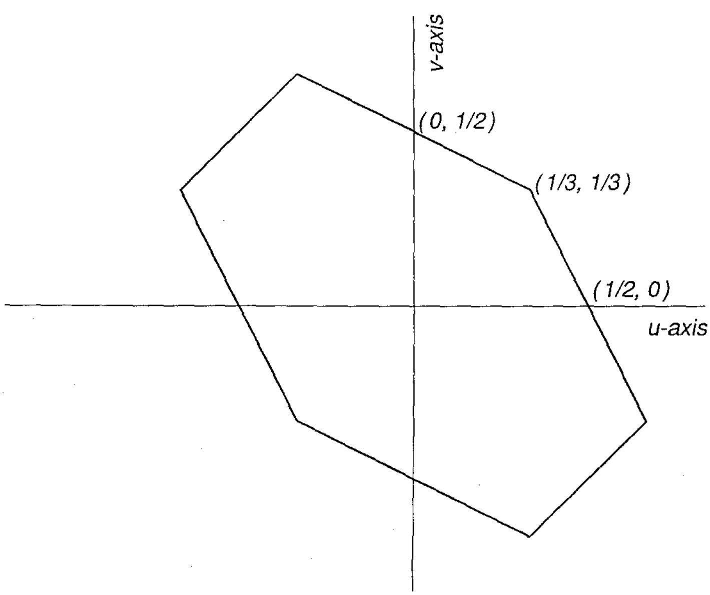


Figure 4.1

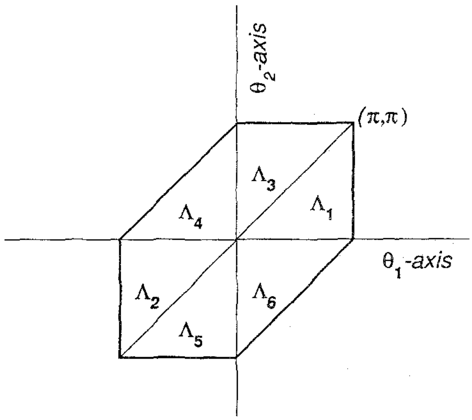


FIGURE 4.2

and $\Delta_2$ is the complement of $\Delta_1$ in $\Delta$. These regions $\Delta_1$ and $\Delta_2$ are chosen so that the estimates below (in Theorem 4.1) hold even for $n=2$. Likewise (see Fig. 4.2), $\Lambda$ is divided into six sub-regions.

It is known that $\bar{\Omega}$ (the closure of $\Omega$) is a *fundamental domain*; i.e., its translates form an essentially disjoint partition of $\mathbf{R}^2/2\pi$ (see [1] and [2]). Further, $\Omega$ and $\bar{\Omega}$ are *invariant under* $\mathbf{A}^*$ as are $\Lambda$ and $\bar{\Lambda}$ *under* $\mathbf{A}$.

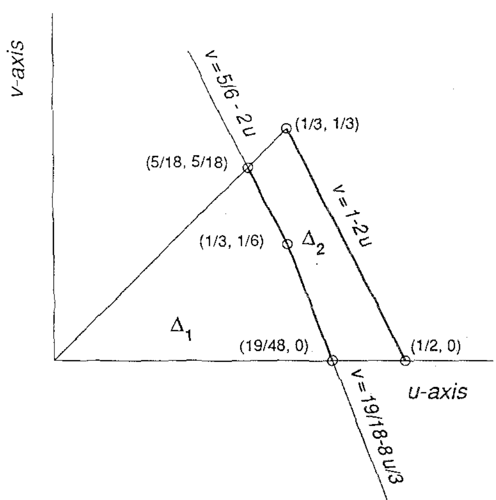


FIGURE 4.3

From (1.2) and (1.3), we see that

$$\begin{aligned} &e^{-i(u\theta_1+v\theta_2)}P_{n,\alpha}(2\pi u, 2\pi v)\\ &\quad= [S(2\pi u)S(2\pi v)S(2\pi(u+v))]^n\\ &\qquad\times\left[1+\sum_{(k,l)\in\mathbf{Z}^2\setminus(0,0)}\left(\frac{u}{u+k}\right)^n\left(\frac{v}{v+l}\right)^n\left(\frac{u+v}{u+v+k+l}\right)^n e^{i(k\theta_1+l\theta_2)}\right]\\ &\quad=: [S(2\pi u)S(2\pi v)S(2\pi(u+v))]^n[Q_{n,\theta}(u,v)], \end{aligned} \tag{4.1}$$

where $S$ is the sinc function defined in (1.4).

This section is arranged as follows. We first show that $P_{n,\alpha}(2\pi u, 2\pi v)$ does not vanish on $\bar{\Omega}$ (and hence in $\mathbf{R}^2$) for $\theta\in\Lambda$ and $n$ even. Then we prove that for all $n$, $P_{n,\alpha}(2\pi u, 2\pi v)$ vanishes somewhere on the boundary of $\Omega$ for all values of $\theta$ on the boundary of $\Lambda$. Finally, with the aid of these results and Theorem 2.1, we obtain a correctness result for $M_{n,\alpha}$ when $n$ is even. This last theorem is the primary objective of both this section and this paper.

First, we make some preliminary observations. It is not hard to see that $\bar{\Omega}=\bigcup_{A\in\mathbf{A}}A^*\tilde{\bar{\Omega}}$ and this, taken together with (3.3) and the invariance of $\Lambda$ under $\mathbf{A}$, allows us to consider only $(u,v)\in\tilde{\bar{\Omega}}$ and $\theta\in\Lambda$ in our subsequent discussion. Moreover, from the definition of $S(t)$ (see 1.4), it is quite clear that $[S(2\pi u)S(2\pi v)S(2\pi(u+v))]^n$ is never zero for $(u,v)\in\tilde{\bar{\Omega}}$. Therefore, the zeroes of $P_{n,\alpha}$ are completely determined by $Q_{n,\theta}(u,v)$. This brings us, then, to our first theorem of this section.

THEOREM 4.1. *Let $n\in\mathbf{N}$ be even and $\theta\in\Lambda$. Then, for $(u,v)\in\tilde{\bar{\Omega}}$, $Q_{n,\theta}(u,v)$ is non-zero.*

*Proof.* To simplify matters, observe that (see (4.1))

$$Q_{n,(\theta_2,\theta_1)}(v,u)=Q_{n,(\theta_1,\theta_2)}(u,v). \tag{4.2}$$

This observation, coupled with the fact that $\tilde{\bar{\Omega}}$ and $\Lambda$ are symmetric about their main diagonals ($u=v$ and $\theta_1=\theta_2$, respectively), permits us to restrict ourselves to $(u,v)\in\Delta$ and $\theta\in\Lambda$ for the duration of the proof. Let us denote the imaginary part of $Q_{n,\theta}$ by

$$I_{n,\theta}(u,v):=\Im[Q_{n,\theta}(u,v)] \tag{4.3}$$

and the imaginary part of $e^{i\theta_2}Q_{n,\theta}(u,v)$ by

$$\tilde{I}_{n,\theta}(u,v):=\Im[e^{i\theta_2}Q_{n,\theta}(u,v)]. \tag{4.4}$$

The proof proceeds in a series of four steps. The actual numerical bounds for the various infinite series that follow were generated by a computer and are accurate to at least four decimal places.

*Step* I.

$$(u, v) \in \Delta, \qquad \theta = (0, 0).$$

In this case, $Q_{n,\theta}(u, v)$ is *strictly positive* by virtue of Theorem 4 on p. 539 of [1].

*Step* II.

$$(u, v) \in \Delta_1, \qquad \theta \in \Delta \setminus (0, 0).$$

From Fig. 4.3, we see that $(u, v) \in \Delta_1$ implies

$$0 \leqslant u \leqslant \tfrac{19}{48}; \qquad 0 \leqslant v \leqslant \tfrac{5}{18}; \qquad \text{and} \qquad 0 \leqslant u+v \leqslant \tfrac{5}{9}. \tag{4.5}$$

We first note that for $0 \leqslant a \leqslant w \leqslant b < 1$, $j \in \mathbf{Z}$, and $n\ (\geqslant 2) \in \mathbf{N}$, one has

$$\left|\frac{w}{w+j}\right|^n \leqslant \left|\frac{b}{b+j}\right|^n, \tag{4.6}$$

a consequence of the fact that the function $g(w) := |w/j+w|$ is non-decreasing on $[a, b]$. Observe, now, that

$$\begin{aligned}
|Q_{n,\theta}(u, v)| &\geqslant 1 - \sum_{(k,l) \in \mathbf{Z}^2 \setminus (0,0)} \left|\frac{u}{u+k}\right|^n \left|\frac{v}{v+l}\right|^n \left|\frac{u+v}{u+v+k+l}\right|^n \\
&\geqslant 1 - \sum_{(k,l) \in \mathbf{Z}^2 \setminus (0,0)} \left|\frac{19}{48k+19}\right|^n \left|\frac{5}{18l+5}\right|^n \left|\frac{5}{9(k+l)+5}\right|^n \\
&\geqslant 1 - \sum_{(k,l) \in \mathbf{Z}^2 \setminus (0,0)} \left|\frac{19}{48k+19}\right|^n \left|\frac{5}{19l+5}\right|^2 \left|\frac{5}{9(k+l)+5}\right|^2 \\
&\geqslant 1 - 0.9939\ldots > 0,
\end{aligned} \tag{4.7}$$

by (4.5), (4.6) and the fact that the summands on the right are less than 1.

So, Step II is complete.

The next two steps (III and IV) are less straightforward, as we shall presently see.

*Step* III.

$$(u, v) \in \Delta_2, \qquad \theta \in [\Delta_1 \cup \Delta_2] \setminus (0, 0).$$

We do the case $\theta \in \Lambda_1$ explicitly. The case $\theta \in \Lambda_2$ can be handled entirely analogously. First, from Fig. 4.3, we have, for all $(u, v) \in \Delta_2$,

$$\tfrac{5}{18} \leqslant u \leqslant \tfrac{1}{2}; \qquad 0 \leqslant v \leqslant \tfrac{1}{3}; \qquad \tfrac{19}{48} \leqslant u+v \leqslant \tfrac{2}{3}; \qquad v \leqslant u. \tag{4.8}$$

Also, from Fig. 4.2, one can see that $\theta \in \Lambda_1$ implies

$$0 \leqslant \theta_2 \leqslant \theta_1 \leqslant \pi. \tag{4.9}$$

In addition to (4.6), we will also use the inequality:

$$\text{For} \quad 0 \leqslant a \leqslant w \leqslant b < 1, \quad j \in \mathbf{Z} \backslash \{0\}, \quad n (\geqslant 2) \in \mathbf{N},$$
$$\left| \frac{1-w}{j+w} \right|^n \leqslant \left| \frac{1-a}{j+a} \right|^n. \tag{4.10}$$

This follows from the observation that the function $h(w) := |1-w/j+w|$ is non-increasing in the interval $[a, b]$. Furthermore, we also have, for $m \in \mathbf{Z}$ and $\phi \in \mathbf{R}$,

$$\begin{gathered} |\cos \phi| \leqslant 1 \\ \left| \frac{\sin m\phi}{\sin \phi} \right| \leqslant |m|. \end{gathered} \tag{4.11}$$

We now show that $Q_{n,\theta}(u, v)$ has no zeroes by showing that $I_{n,\theta}(u, v)$ has none. By virtue of (4.1), we have

$$\begin{aligned} I_{n,\theta}(u, v) &= \sum_{(k,l) \in \mathbf{Z}^2} \left( \frac{u}{u+k} \right)^n \left( \frac{v}{v+l} \right)^n \left( \frac{u+v}{u+v+k+l} \right)^n \sin(k\theta_1 + l\theta_2) \\ &= \sum_{k+l \neq 0} \left( \frac{u}{u+k} \right)^n \left( \frac{v}{v+l} \right)^n \left( \frac{u+v}{u+v+k+l} \right)^n \sin k\theta_1 \cos l\theta_2 \\ &\quad + \sum_{k+l \neq 0} \left( \frac{u}{u+k} \right)^n \left( \frac{v}{v+l} \right)^n \left( \frac{u+v}{u+v+k+l} \right)^n \cos k\theta_1 \sin l\theta_2 \\ &\quad + \sum_{k+l=0} \left( \frac{u}{u+k} \right)^n \left( \frac{v}{v+l} \right)^n \left( \frac{u+v}{u+v+k+l} \right)^n \sin(k\theta_1 + l\theta_2) \\ &=: S' + S'' + S'''. \end{aligned} \tag{4.12}$$

We will handle each of these three sums separately by isolating an appropriate dominant term from each of them and factoring it out. If $\tau(k, l)$ represents the $(k, l)$th term in (4.12), then the dominant terms for $S'$, $S''$, and $S'''$ are $\tau(-1, 0)$, $\tau(0, -1)$, and $\tau(1, -1) + \tau(-1, 1)$, respectively. Thus, we have

$$S' = -\left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin\theta_1$$
$$\times\left[1-\sum{}'\left(\frac{1-u}{u+k}\right)^n\left(\frac{v}{v+l}\right)^n\left(\frac{1-u-v}{u+v+k+l}\right)^n\frac{\sin k\theta_1}{\sin\theta_1}\cos l\theta_2\right], \tag{4.13}$$

$$S'' = -\left(\frac{v}{1-v}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin\theta_2$$
$$\times\left[1-\sum{}''\left(\frac{u}{u+k}\right)^n\left(\frac{1-v}{v+l}\right)^n\left(\frac{1-u-v}{u+v+k+l}\right)^n\frac{\sin l\theta_2}{\sin\theta_2}\cos k\theta_1\right], \tag{4.14}$$

and

$$S''' = -(-1)^n\left(\frac{u}{1-u}\right)^n\left(\frac{v}{1+v}\right)^n \sin(\theta_1-\theta_2)$$
$$\times\left[1-\left(\frac{1-u}{1-v}\right)^n\left(\frac{1+v}{1+u}\right)^n\right]$$
$$\times\left[1+\sum_{r=2}^{\infty}\left(\frac{1-u}{r-u}\right)^n\left(\frac{1+v}{r+v}\right)^n a_r(u,v,\theta_1,\theta_2)\right], \tag{4.15}$$

where $\sum'$ denotes the sum over

$$(k,l)\in\mathbf{Z}^2 \quad \text{with} \quad k+l\neq 0, \quad (k,l)\neq(-1,0), \quad \text{and} \quad k\neq 0,$$

$\sum''$ represents the sum taken over

$$(k,l)\in\mathbf{Z}^2 \quad \text{with} \quad k+l\neq 0, \quad (k,l)\neq(0,-1), \quad \text{and} \quad l\neq 0,$$

and

$$a_r(u,v,\theta_1,\theta_2) := \left[\frac{1-\left(\dfrac{r-u}{r-v}\right)^n\left(\dfrac{r+v}{r+u}\right)^n}{1-\left(\dfrac{1-u}{1-v}\right)^n\left(\dfrac{1+v}{1+u}\right)^n}\right]\frac{\sin r(\theta_1-\theta_2)}{\sin(\theta_1-\theta_2)}. \tag{4.16}$$

Let us consider the sums that occur in (4.13), (4.14), and (4.15) separately. Using the inequalities (4.6), (4.8), (4.10), and (4.11) repeatedly, we find that for (4.13),

$$1-\sum{}'\left(\frac{1-u}{u+k}\right)^n\left(\frac{v}{v+l}\right)^n\left(\frac{1-u-v}{u+v+k+l}\right)^n\frac{\sin k\theta_1}{\sin\theta_1}\cos l\theta_2$$
$$\geqslant 1-\sum{}'\left|\frac{1-u}{u+k}\right|^n\left|\frac{v}{v+l}\right|^n\left|\frac{1-u-v}{u+v+k+l}\right|^n\left|\frac{\sin k\theta_1}{\sin\theta_1}\right|\,|\cos l\theta_2|$$

$$\geqslant 1-\sum{}' \left|\frac{13}{18k+5}\right|^n \left|\frac{1}{3l+1}\right|^n \left|\frac{29}{48(k+l)+19}\right|^n |k|$$

$$\geqslant 1-\sum{}' \left|\frac{13}{18k+5}\right|^2 \left|\frac{1}{3l+1}\right|^2 \left|\frac{29}{48(k+l)+19}\right|^2 |k|$$

$$\geqslant 1-0.3342\ldots>0 \tag{4.17}$$

and similarly, for (4.14),

$$1-\sum{}'' \left(\frac{u}{u+k}\right)^n \left(\frac{1-v}{v+l}\right)^n \left(\frac{1-u-v}{u+v+k+l}\right)^n \frac{\sin l\theta_2}{\sin \theta_2} \cos k\theta_1$$

$$\geqslant 1-\sum{}'' \left|\frac{u}{u+k}\right|^n \left|\frac{1-v}{v+l}\right|^n \left|\frac{1-u-v}{u+v+k+l}\right|^n \left|\frac{\sin l\theta_2}{\sin \theta_2}\right| |\cos k\theta_2|$$

$$\geqslant 1-\sum{}'' \left|\frac{1}{2k+1}\right|^n \left|\frac{1}{l}\right|^n \left|\frac{29}{48(k+l)+19}\right|^n |l|$$

$$\geqslant 1-\sum{}'' \left|\frac{1}{2k+1}\right|^2 \left|\frac{1}{l}\right|^2 \left|\frac{29}{48(k+l)+19}\right|^2 |l|$$

$$\geqslant 1-0.9821\ldots>0. \tag{4.18}$$

To estimate the sum in (4.15) from below, we also need the following: for $r \in \mathbf{N}$,

$$0 \leqslant v \leqslant u \Rightarrow 0 \leqslant \left(\frac{1-u}{1-v}\right)\left(\frac{1+v}{1+u}\right) \leqslant \left(\frac{r-u}{r-v}\right)\left(\frac{r+v}{r+u}\right) \leqslant 1. \tag{4.19}$$

Inequalities (4.11) and (4.19) ensure that (cf. (4.16)), for $r \in \mathbf{N}$,

$$|a_r(u, v, \theta_1, \theta_2)| \leqslant 1 \cdot r. \tag{4.20}$$

Consequently, one has

$$1+\sum_{r=2}^{\infty} \left(\frac{1-u}{r-u}\right)^n \left(\frac{1+v}{r+v}\right)^n a_r(u, v, \theta_1, \theta_2)$$

$$\geqslant 1-\sum_{r=2}^{\infty} \left|\frac{1-u}{r-u}\right|^n \left|\frac{1+v}{r+v}\right|^n |a_r(u, v, \theta_1, \theta_2)|$$

$$\geqslant 1-\sum_{r=2}^{\infty} \left(\frac{1}{r}\right)^n \left(\frac{4}{3r+1}\right)^n r$$

$$\geqslant 1-\sum_{r=2}^{\infty} \left(\frac{1}{r}\right)^2 \left(\frac{4}{3r+1}\right)^2 r$$

$$\geqslant 1-0.28\ldots>0. \tag{4.21}$$

From (4.17), (4.18), and (4.21), we conclude that the signs of $S'$, $S''$, and $S'''$ *coincide with those of their respective dominant terms* and that *their zeroes are precisely those of these dominant terms.* But, from (4.9), (4.19), and the fact that $n$ is even, one can see that *all of these three dominant terms are of the same sign* and that *they do not vanish simultaneously.* This shows that $I_{n,\theta}(u, v)$ is non-zero and finishes Step III.

*Step* IV. In this final step, we exhaust the only remaining case; namely,

$$(u, v) \in \Delta_2, \qquad \theta \in \left[\bigcup_{i=3}^{6} \Lambda_i\right] \setminus (0, 0).$$

Note that the estimates for $u$, $v$, and $u+v$ are still given by (4.8) as in the previous step.

Here we will prove that $Q_{n,\theta}(u, v)$ has no zeroes by showing that $e^{i\theta_2} Q_{n,\theta}(u, v)$ does not have any; the latter, in turn, will be achieved by demonstrating the absence of zeroes for $\tilde{I}_{n,\theta}(u, v)$. It will become quite apparent that the *modus operandi* for doing so is similar to that in Step III.

The fist part of the proof involves some tedious, albeit unavoidable calculations. We commence by writing

$$\begin{aligned}
\tilde{I}_{n,\theta}(u, v) &= \sin \theta_2 + \sum_{(k,l) \in \mathbf{Z}^2 \setminus (0,0)} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \\
&\quad \times \left(\frac{u+v}{u+v+k+l}\right)^n \sin(k\theta_1 + (l+1)\theta_2) \\
&= \sin \theta_2 + \left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin(\theta_2 - \theta_1) \\
&\quad + \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \\
&\quad \times \left(\frac{u+v}{u+v+k+l}\right)^n \cos k\theta_1 \sin(l+1)\theta_2 \\
&\quad + \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \\
&\quad \times \left(\frac{u+v}{u+v+k+l}\right)^n \sin k\theta_1 \cos(l+1)\theta_2,
\end{aligned} \tag{4.22}$$

where $\sum'''$ stands for the sum over

$$(k, l) \in \mathbf{Z}^2; \qquad (k, l) \neq (0, 0), \quad (k, l) \neq (-1, 0).$$

The last sum in (4.22) can be recast by writing $\theta_1 = \theta_2 - (\theta_2 - \theta_1)$; the resulting expression for $\tilde{I}_{n,\theta}$ is

$$\begin{aligned}
\tilde{I}_{n,\theta}(u, v) = \sin \theta_2 + \left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin(\theta_2 - \theta_1) \\
+ \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
\times \cos k\theta_1 \sin(l+1)\theta_2 \\
+ \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
\times \sin k\theta_2 \cos k(\theta_2 - \theta_1) \cos(l+1)\theta_2 \\
- \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
\times \sin k(\theta_2 - \theta_1) \cos k\theta_2 \cos(l+1)\theta_2. \qquad (4.23)
\end{aligned}$$

The first two summands in (4.23) constitute the dominant terms in the expression for $\tilde{I}_{n,\theta}(u, v)$. Accordingly, we assign the appropriate sums in (4.23) to each of these (dominant) terms and then factor them out. More precisely, we write (4.23) as

$$\begin{aligned}
\tilde{I}_{n,\theta}(u, v) = \sin \theta_2 \Bigg[ 1 + \sum_{l \neq -1}{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
\times \frac{\sin(l+1)\theta_2}{\sin \theta_2} \cos k\theta_1 \\
+ \sum_{k \neq 0}{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
\times \frac{\sin k\theta_2}{\sin \theta_2} \cos k(\theta_2 - \theta_1) \cos(l+1)\theta_2 \Bigg] \\
+ \left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin(\theta_2 - \theta_1) \\
\times \Bigg[ 1 - \sum{}' \left(\frac{1-u}{k+u}\right)^n \left(\frac{v}{l+v}\right)^n \left(\frac{1-u-v}{k-l+u+v}\right)^n \\
\times \frac{\sin k(\theta_2 - \theta_1)}{\sin(\theta_2 - \theta_1)} \cos k\theta_2 \cos(l+1)\theta_2
\end{aligned}$$

$$-\left(\frac{v}{u+v}\right)^n \sum_{\substack{k+l=0\\ k\neq 0}} \left(\frac{1-u}{k+u}\right)^n \left(\frac{1-u-v}{l+v}\right)^n$$

$$\left.\times \frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)} \cos k\theta_2 \cos(l+1)\theta_2\right] =: T' + T'', \tag{4.24}$$

where $\sum'$ and $\sum'''$ are the same as before.

Just as in the previous step, we estimate, from below, the sums that are featured in $T'$ and $T''$ individually and show that they are bounded away from zero. Again making use of the inequalities given by (4.6), (4.8), (4.10), and (4.11), we have for $T'$,

$$\begin{aligned}
1 &+ \sum_{l\neq -1}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
&\quad\times \frac{\sin(l+1)\theta_2}{\sin\theta_2} \cos k\theta_1 \\
&\quad+ \sum_{k\neq 0}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \\
&\quad\times \frac{\sin k\theta_2}{\sin\theta_2} \cos k(\theta_2-\theta_1) \cos(l+1)\theta_2 \\
&\geqslant 1 - \sum_{l\neq -1}''' \left|\frac{u}{u+k}\right|^n \left|\frac{v}{v+l}\right|^n \left|\frac{u+v}{u+v+k+l}\right|^n \left|\frac{\sin(l+1)\theta_2}{\sin\theta_2}\right| |\cos k\theta_1| \\
&\quad- \sum_{k\neq 0}''' \left|\frac{u}{u+k}\right|^n \left|\frac{v}{v+l}\right|^n \left|\frac{u+v}{u+v+k+l}\right|^n \left|\frac{\sin k\theta_2}{\sin\theta_2}\right| \\
&\quad\times |\cos k(\theta_2-\theta_1)\cos(l+1)\theta_2| \\
&\geqslant 1 - \sum_{l\neq -1}''' \left|\frac{1}{2k+1}\right|^n \left|\frac{1}{3l+1}\right|^n \left|\frac{2}{3(k+l)+2}\right|^n |l+1| \\
&\quad- \sum_{k\neq 0}''' \left|\frac{1}{2k+1}\right|^n \left|\frac{1}{3l+1}\right|^n \left|\frac{2}{3(k+l)+2}\right|^n |k| \\
&\geqslant 1 - \sum_{l\neq -1}''' \left|\frac{1}{2k+1}\right|^2 \left|\frac{1}{3l+1}\right|^2 \left|\frac{2}{3(k+l)+2}\right|^2 |l+1| \\
&\quad- \sum_{k\neq 0}''' \left|\frac{1}{2k+1}\right|^2 \left|\frac{1}{3l+1}\right|^2 \left|\frac{2}{3(k+l)+2}\right|^2 |k| \\
&\geqslant 1 - 0.9812\ldots > 0.
\end{aligned} \tag{4.25}$$

Turning to $T''$, we use the additional fact that for $n$ $(\geqslant 2) \in \mathbf{N}$,

$$0 \leqslant v \leqslant u \Rightarrow 0 \leqslant \left(\frac{v}{u+v}\right)^n \leqslant \left(\frac{1}{2}\right)^n \leqslant \frac{1}{4}$$

to deduce that

$$\begin{aligned}
1 - \sum{}' & \left(\frac{1-u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{1-u-v}{u+v+k+l}\right)^n \\
& \times \frac{\sin k(\theta_2 - \theta_1)}{\sin(\theta_2 - \theta_1)} \cos k\theta_2 \cos(l+1)\theta_2 \\
& - \left(\frac{v}{u+v}\right)^n \sum_{\substack{k+l=0 \\ k \neq 0}} \left(\frac{1-u}{u+k}\right)^n \left(\frac{1-u-v}{l+v}\right)^n \\
& \times \frac{\sin k(\theta_2 - \theta_1)}{\sin(\theta_2 - \theta_1)} \cos k\theta_2 \cos(l+1)\theta_2 \\
\geqslant 1 - \sum{}' & \left|\frac{1-u}{u+k}\right|^n \left|\frac{v}{v+l}\right|^n \left|\frac{1-u-v}{u+v+k+l}\right|^n \\
& \times \left|\frac{\sin k(\theta_2 - \theta_1)}{\sin(\theta_2 - \theta_1)}\right| |\cos k\theta_2 \cos(l+1)\theta_2| \\
& - \left|\frac{v}{u+v}\right|^n \sum_{\substack{k+l=0 \\ k \neq 0}} \left|\frac{1-u}{u+k}\right|^n \left|\frac{1-v}{v+l}\right|^n \\
& \times \left|\frac{\sin k(\theta_2 - \theta_1)}{\sin(\theta_2 - \theta_1)}\right| |\cos k\theta_2 \cos(l+1)\theta_2| \\
\geqslant 1 - \sum{}' & \left|\frac{13}{18k+5}\right|^n \left|\frac{1}{3l+1}\right|^n \left|\frac{29}{48(k+l)+19}\right|^n |k| \\
& - \left(\frac{1}{2}\right)^n \sum_{\substack{k+l=0 \\ k \neq 0}} \left|\frac{13}{18k+5}\right|^n \left|\frac{1}{l}\right|^n |k| \\
\geqslant 1 - \sum{}' & \left|\frac{13}{18k+5}\right|^2 \left|\frac{1}{3l+1}\right|^2 \left|\frac{29}{48(k+l)+19}\right|^2 |k| \\
& - \frac{1}{4} \sum_{\substack{k+l=0 \\ k \neq 0}} \left|\frac{13}{18k+5}\right|^2 \left|\frac{1}{l}\right|^2 |k| \\
\geqslant 1 - 0.4854\ldots > 0. &
\end{aligned} \tag{4.26}$$

From (4.25) and (4.26), it is clear that $T'$ and $T''$ *have the same signs as their respective dominant terms* and that *their zeroes are those of these dominant terms.* Now, for $(u, v) \in \Delta_2$ and $\theta \in [\bigcup_{i=3}^{6} \Delta_i] \backslash (0, 0)$, these dominant terms are *both of the same sign* and are *never zero simultaneously*. As a result, it follows readily that $\tilde{T}_{n,\theta}(u, v)$ is non-zero as desired. This concludes Step IV and with it the proof of the theorem in its entirety.

*Remark.* Since the case $n=1$ can be handled directly, the various estimates in the proof of Theorem 4.1 which used the fact that $n \geqslant 2$ do not, in fact, rely on the evenness of $n$. The only stage in which the evenness is used is in Step III.

Our next result which holds true for all (*not just even*) $n$, serves to locate the zeroes of $P_{n,\alpha}(x)$.

THEOREM 4.2. *Let* $n \in \mathbf{N}$ *and* $0 \leqslant \gamma \leqslant 1/2$. *Then the following hold*:

(i) $P_{n,(1/2,\gamma)}(\pi, 0) = 0 = P_{n,(-1/2,-\gamma)}(\pi, 0)$.

(ii) $P_{n,(\gamma, 1/2)}(0, \pi) = 0 = P_{n,(-\gamma,-1/2)}(0, \pi)$.

(iii) (a) $P_{n,(\gamma,1/2)}(0, -\pi) = 0 = P_{n,(-\gamma,-1/2)}(0, -\pi)$.

(b) $P_{n,(-1/2+\gamma,\gamma)}(-\pi, \pi) = 0 = P_{n,(1/2-\gamma,-\gamma)}(-\pi, \pi)$.

(iv) (a) $P_{n,(1/2,\gamma)}(-\pi, 0) = 0 = P_{n,(-1/2,-\gamma)}(-\pi, 0)$.

(b) $P_{n,(\gamma,\gamma-1/2)}(\pi, -\pi) = 0 = P_{n,(-\gamma,-\gamma+1/2)}(\pi, -\pi)$.

*Proof.* It suffices to prove (i) and (ii). Assertions (iii) and (iv) follow from (i) and (ii), respectively, by invoking the symmetry conditions stated in (3.3).

To prove (i), we set $v=0$ in (4.1). The resulting expression depends only on $\theta_1$:

$$e^{-iu\theta_1} P_{n,\alpha}(2\pi u, 0) = [S(2\pi u)]^{2n} \left[1 + \sum_{k \in \mathbf{Z} \backslash \{0\}} \left(\frac{u}{u+k}\right)^{2n} e^{ik\theta_1}\right]. \quad (4.27)$$

Since the RHS of (4.27) is zero if and only if [5, Theorem 2.2]

$$u = 1/2 \qquad \text{and} \qquad \theta_1 = \pm\pi,$$

(i) is proved.

Similarly, setting $u=0$ in (4.1) yields (ii).

We now proceed to our main result, which is

THEOREM 4.3. *Let $n \in \mathbf{N}$ be even and $2\pi\alpha \in \bar{\Lambda}$. Then cardinal interpolation with $M_{n,\alpha}$ is correct if and only if $2\pi\alpha \in \Lambda$.*

*Proof.* As we remarked earlier, it is manifest from (4.1) and the definition of $S(t)$ that $P_{n,\alpha}(2\pi u, 2\pi v)$ is zero if and only if $Q_{n,\theta}(u, v)$ is zero. The desired result now follows from Theorems 4.1 and 4.2 via Theorem 2.1.

*Remark.* K. Jetter (in [3]) has indicated that the above result has been obtained independently by J. Stöckler with $M_{n,\alpha}$ replaced by

$$M_{(k,l,m),\alpha}; \qquad 1 \leqslant k, l, m \leqslant 2, \quad k+l+m \leqslant 5.$$

Here $M_{(k,l,m)}$ denotes the bivariate box spline corresponding to the three directions $(1, 0)$, $(0, 1)$, and $(1, 1)$ which occur with multiplicities $k$, $l$, and $m$ respectively. However, it should be pointed out that neither Stöckler's result nor ours can be deduced from the other.

## 5. CONCLUDING REMARKS

A scrutiny of the proof of Theorem 4.1 reveals that a crucial point in the argument in Step III (the only place where the evenness of $n$ is used) is the fact that *for $\theta \in \Lambda_1$ (resp. $\Lambda_2$), $I_{n,\theta}(u, v)$ does not change sign in $\Delta_2$*. However, one can show that *this is no longer true when n is odd.* Moreover, the situation, in this case, cannot be remedied by rotating $Q_{n,\theta}(u, v)$ by a suitable angle (as done in Step IV). As a result, our method of proof fails for splines with odd multiplicities. Nevertheless, we believe that Theorem 4.3 remains valid in this situation as well.

One other remark also seems to be in order here. Suppose that $\alpha$ is chosen along any of the three mesh directions, i.e., $\alpha$ is of the form $(\gamma, 0)$, $(0, \gamma)$, or $(\gamma, \gamma)$, where $-\frac{1}{2} \leqslant \gamma \leqslant \frac{1}{2}$. Then, from Theorem 4.2 and the proof of Theorem 4.1 (specifically, Steps I, II, and IV) and the fact that the set $\{(\gamma, 0), (0, \gamma), (\gamma, \gamma)\}$ is invariant under $\mathbf{A}$, it follows that *for any n*, cardinal interpolation with $M_{n,\alpha}$ is correct if and only if $-\pi < 2\pi\gamma < \pi$.

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